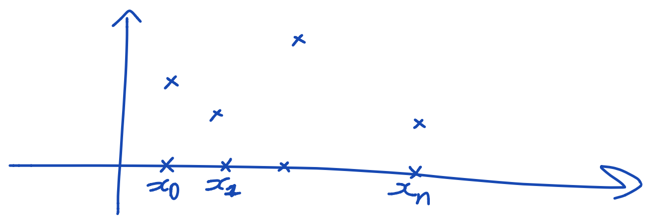
# Polynomial Interpolation

**Goal:** Given a set of $n+1$ data points $(x_i, y_i)$




we seek a polynomial $P$ with lowest possible degree so that $P(x_i) = y_i$

We say that $P$ interpolates the data.

**Theorem:** If $x_0, x_1, \ldots, x_n$ are distinct real numbers then for arbitrary $y_0, \ldots, y_n$ $\exists$ a unique polynomial $P_n$ of degree $n$ or less s.t.

$$P_n(x_i) = y_i \text{ for } i = 0, 1, \ldots n$$

**Proof:** (1) Existence:

We'll do it by induction

- for $n=0$ we can always find $P_0$ s.t. $P_0(x_0) = y_0$    const. ft.

- Suppose $P_{k-1}$ satisfies $P_{k-1}(x_i) = y_i$ for $i = 0, 1, \ldots, k-1$

  Let $P_k(x) = P_{k-1}(x) + C\underbrace{(x-x_0)(x-x_1)\cdots(x-x_{k-1})}_{\text{degree } \leq k}$

  To find $C$ we just solve $P_k(x_k) = y_k$

  so $\underbrace{P_k(x_k)}_{y_k} = \underbrace{P_{k-1}(x_k)}_{\text{known bec } P_{k-1} \text{ is known}} + C\underbrace{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})}_{\text{all} \neq 0 \text{ bec } x_i\text{'s distinct}}$ $\Rightarrow$ can solve!

we also note that $P_k(x_i) = P_{k-1}(x_i) = y_i$ for $i = 0, \ldots, k-1$

so $P_k$ interpolates the data. $(x_i, y_i)$, $i = 0, \ldots, k$

Uniqueness: Suppose $\exists q_n$ that interpolates the data

then $q_n(x_i) - P_n(x_i) = 0$ for $i = 0, \ldots, n$

$$\Rightarrow \underbrace{(q_n - P_n)}_{\text{poly of degree} \le n}\underbrace{(x_i) = 0}_{n+1 \text{ of these}}$$

with $n+1$ zeros $\Rightarrow q_n - P_n = 0$ ☒

## Newton form of the interpolating polynomial

From the proof: $P_k(x) = P_{k-1}(x) + c_k(x - x_0) \cdots (x - x_{k-1})$

$$= \cdots$$

$$= c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots$$

$$+ c_k(x - x_0) \ldots (x - x_{k-1})$$

In short form: $\boxed{P_k(x) = \sum_{i=0}^{k-1} c_i \prod_{j=0}^{i-1} (x - x_j)}$

interp. polynomials in Newton form.

# Lagrange form of interpolating polynomial

We can write our polynomial as

$$P_n(x) = \sum_{k=0}^{n} y_k \, \ell_k(x)$$

$\rightarrow$ polynomials depending on ($\uparrow$ degree $\leq n$) the nodes $x_0 — x_n$, not the ordinates $y_0 — y_n$

Specifically we (may choose) $\ell_k(x_i) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$

$$= \delta_{ik}$$

This can be done, e.g., by setting

$$\ell_0(x) = c \prod_{i=1}^{n} (x - x_i) \Rightarrow c = \frac{\overbrace{\ell_0(x_0)}^{1}}{\prod_{i=1}^{n}(x_0 - x_i)} = \prod_{i=1}^{n}(x_0 - x_i)^{-1}$$

$$\Rightarrow \quad \ell_0(x) = \prod_{i=1}^{n} \frac{x - x_i}{x_0 - x_i}$$

Similarly $\quad \ell_j(x) = \prod_{\substack{i \neq j \\ i=0}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$

cardinal functions

$$P_n(x) = \sum_{i=0}^{n} y_i \, \ell_i(x)$$

Lagrange form

**Example** Given

| $x$ | 5 | $-7$ | $-6$ | 0 |
|---|---|---|---|---|
| $y$ | 1 | $-23$ | $-54$ | 954 |

Find the cardinal functions and the Lagrange form of the interpolating polynomial

$$\ell_0(x) = \frac{x-(-7)}{5-(-7)} \cdot \frac{x-(-6)}{5-(-6)} \cdot \frac{x-0}{5-0} = \frac{(x+6)(x+7)x}{660}$$

$$\ell_1(x) = \frac{x-5}{-7-5} \cdot \frac{x-(-6)}{-7-(-6)} \cdot \frac{x-0}{-7-0} = \frac{(x-5)(x+6)x}{-84}$$

$\ell_2(x) = \ldots$

$\ell_3(x) = \ldots$

$\Rightarrow P_3(x) = \ell_0(x) - 23\,\ell_1(x) - 54\,\ell_2(x) - 954\,\ell_3(x)$

**Theorem** (error in polynomial interp)

Let $f \in C^{n+1}[a,b]$ & $P$ be a poly. of degree $\leq n$ that interpolates $f$ at $n+1$ distinct pts $x_0, x_1, \ldots, x_n \in [a,b]$.

Then $\forall x \in [a,b]$, $\exists\, \xi_x \in (a,b)$ s.t.

$$f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

(A)

Exercise: read the proof.

# Divided Differences

Recall: $\boxed{P_k(x) = \sum_{i=0}^{k-1} c_i \prod_{j=0}^{i-1}(x-x_j)}$

interp. polynomials in Newton form.

If we call $q_0(x) = 1$

$q_1(x) = x - x_0$

$q_2(x) = (x-x_1)(x-x_0)$

$\vdots$

$q_n(x) = \prod_{i=0}^{n-1}(x-x_i)$

then $P_n(x) = \sum_{j=0}^{n} c_j q_j(x)$

Recall: the $c_j$'s are what we need to find.

We have n+1 linear equations:

$$\sum_{j=0}^{n} c_j \underbrace{q_j(x_i)}_{a_{ij}} = \underbrace{f(x_i)}_{y_i}, \quad i = 0, 1, \ldots, n+1$$

In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1-x_0 & 0 & \cdots & 0 \\ 1 & x_1-x_0 & (x_2-x_0)(x_2-x_1) & \cdots & 0 \\ \vdots & & & \ddots & \\ & & & & \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

So: $c_0$ depends on $y_0$, we write $c_0 = f[x_0]$

$c_1$ depends on $y_0, y_1$, we write $c_1 = f[x_0, x_1]$

$\vdots$

$c_n$ depends on $y_0, y_1, \text{—}, y_n$, we write $c_n = f[x_0, x_1, \text{—}, x_n]$

old notation

Let's compute a couple of these:

- $f[x_0] = f(x_0) \quad (= c_0)$
- $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (= c_1)$

& $P_1(x) = f(x_0) + \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$

Theorem: $f[x_0, x_1 \text{—}, x_n] = \dfrac{f[x_1, \text{—}, x_n] - f[x_0, \text{—}, x_{n-1}]}{x_n - x_0}$

proof: exercise/read it

So $\cdot\ f[x_0, x_1, x_2] = \dfrac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$

This makes it easy to compute the f[ ]'s using a divided differences table

| | | | | |
|---|---|---|---|---|
| $x_0$ | $f[x_0]$ | $f[x_0, x_1]$ | $f[x_0, x_1, x_2]$ | $f[x_0,x_1,x_2,x_3]$ |
| $x_1$ | $f[x_1]$ | $f[x_1, x_2]$ | $f[x_1, x_2, x_3]$ | |
| $x_2$ | $f[x_2]$ | $f[x_2, x_3]$ | | |
| $x_3$ | $f[x_3]$ | | | |

e.g. Use divided differences to find the Newton interpolating polynomial for

| $x$ | 3 | 1 | 5 | 6 |
|---|---|---|---|---|
| $f(x)$ | 1 | $-3$ | 2 | 4 |

Sol'n: Step 1 (table)

| $x$ | $f[x]$ | | | |
|---|---|---|---|---|
| 3 | 1 | $\dfrac{-3-1}{1-3} = 2$ | $\dfrac{5/4 - 2}{5-3} = -3/8$ | $\dfrac{3/20 - (-3/8)}{6-3} = \dfrac{7}{40}$ |
| 1 | $-3$ | $\dfrac{2-(-3)}{5-1} = 5/4$ | $(2 - 5/4)/5 = 3/20$ | |
| 5 | 2 | $\dfrac{4-2}{6-5} = 2$ | | |
| 6 | 4 | | | |

Step 2: write the polynomial

$$P_3(x) = 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) + \frac{7}{40}(x-3)(x-1)(x-5)$$

Property: If $(z_0, \dots, z_n)$ is a permutation of $(x_0, \dots, x_n)$ then

$$f[x_0, \dots, x_n] = f[z_0, \dots, z_n]$$

both are the coef of $x^n$ in the interp. Poly.

## Hermite Interpolation

Want to interpolate not only the function, but also its derivatives

Setup: at each $x_i$ we are given $P^{(j)}(x_i)$ for $0 \le j \le k_i - 1$

e.g. Find a polynomial $P$ with $P(0) = 0$, $P'(0) = 1$ & $P(1) = 1$

<u>In general</u>: $P^{(j)}(x_i) = c_{ij}$ $(0 \le j \le k_i - 1,\ 0 \le i \le n)$
where $\sum_{i=0}^{n} k_i = m+1$

So, we have $m+1$ conditions $\Rightarrow$ reasonable to look for an $m^{th}$ degree polynomial.

**Theorem**: $\exists$ a unique polynomial of degree at most $m$ satisfying

$$P^{(j)}(x_i) = c_{ij} \quad (0 \le j \le k_i - 1,\ 0 \le i \le n)$$
where $\sum_{i=0}^{n} k_i = m+1$

<u>Example</u>: Hermite interpolation with <u>only one</u> <u>node</u>.

Given: $P^{(j)}(x_0)$ for $0 \le j \le k$

$$\Rightarrow P(x) = P(x_0) + P'(x_0)(x-x_0) + P''(x_0)(x-x_0)^2/2! + \cdots + P^{(k)}(x_0)\frac{(x-x_0)^k}{k!}$$

Taylor polynomial!

# Newton Divided Difference Method

Extensions to some Hermite interpolation

Example: Use extended Newton divided diff to find a polynomial satisfying

$$P(1)=2,\ P'(1)=3,\ P(2)=6,\ P'(2)=7,\ P''(2)=8$$

Will come back to it.

Method:

$x_0, x_0, x_1, x_1, x_1$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_0$ | $f[x_0]$ | $f[x_0,x_0]$ | $f[x_0,x_0,x_1]$ | $f[x_0,x_0,x_1,x_1]$ | $f[-]$ |
| $x_0$ | $f[x_0]$ | $f[x_0,x_1]$ | $f[x_0,x_1,x_1]$ | $f[x_0,x_1,x_1,x_1]$ | |
| $x_1$ | $f[x_1]$ | $f[x_1,x_1]$ | $f[x_1,x_1,x_1]$ | | |
| $x_1$ | $f[x_1]$ | | | | |
| $x_1$ | $f[x_1]$ | | | | |

$f[x_0,x_0]$

But what is $f[x_0,x_0]$ ? $\lim_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = \boxed{f'(x_0)}$

$$f[x_0,x_0,x_0] = ?$$

Apply Theorem (A): $\exists\, \xi = x_0$ : $\boxed{f[x_0,x_0,x_0] = \dfrac{f''(x_0)}{2}}$

Similarly $\boxed{f[\underbrace{x_0, \text{———} x_0}_{k+1 \text{ time}}] = \dfrac{f^{(k)}(x_0)}{k!}}$

OK back to the example

$$P(1)=2,\ P'(1)=3,\ P(2)=6,\ P'(2)=7,\ P''(2)=8$$

| $x$ | $f(x)$ | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | $\frac{4-3}{1}=1$ | $\frac{3-1}{1} \to 2$ | $\frac{1-2}{2-1} \to -1$ |
| 1 | 2 | $\frac{6-2}{2-1}=4$ | $\frac{7-4}{1}=3$ | $\frac{4-3}{1} \to 1$ | |
| 2 | 6 | 7 | $\frac{8}{2!}$ | | |
| 2 | 6 | | | | |
| 2 | 6 | | | | |

Now we use the top row to write $p(x)$

$$P(x) = 2 + 3(x-1) + 1(x-1)^2 + 2(x-1)^2(x-2) - (x-1)^2(x-2)^2$$

Check: $p(1) = 2$ ✓

$p'(1) = 3$ ✓

$p(2) = 2 + 3(2-1) + 1(2-1) + 0 = 6$ ✓

$p'(2) = 3 + 2(2-1) + 2(2-1)^2 = 7$ ✓

$p''(2) = 2 + \dots = 8$ ✓